MOLDAGALI YERKEBULAN OMIRGALIULY

«SOLVABILITY AND REGULARITY CONDITIONS FOR DIFFERENTIAL EQUATIONS WITH UNBOUNDED COEFFICIENTS»

ABSTRACT

of the dissertation submitted for the degree of doctor of philosophy (PhD) in the educational program 8D05401 – "Mathematics"

General characteristics of the work. The dissertation is devoted to the issues of solvability conditions for fourth-order differential equations defined in an unbounded domain, and the maximal regularity of their solutions.

Relevance of the topic. It is known from the works of V.I. Erofeev, V.V. Kazhaev, N.P. Semerikova, L.A. Ostrovsky, I.A. Potapov and other authors that various practical problems lead to the solution of fourth-order differential equations with variable coefficients. A simple biharmonic equation with small terms plays an important role in elasticity theory, the mechanics of elastic plates and in the study of slow viscous fluid flow. However, it represents a very special case of fourth-order differential equations. The issues of the existence and uniqueness of solutions to boundary value problems for linear and nonlinear fourth-order differential equations with regular coefficients have been widely studied in the literature. However, some problems in stochastic analysis, oscillation theory, biology and financial mathematics lead to differential equations with unbounded intervals and intermediate coefficients. The solvability conditions for such equations and the qualitative properties of the solution depend mainly on the change in the coefficients in the neighborhood of infinitely distant points and their interrelationships. A fourth-order singular differential equation is sometimes used as a regularizer in the study of lower-order equations such as the Korteweg-de Vries equations or reaction-diffusion equations.

The first three sections of the paper consider binomial equations of the form

$$y^{(4)} + p_j(x)y^{(j)} = F(x), \ \ j = 1,2,3, \qquad (1)$$

where $x \in \mathbb{R} = (-\infty, \infty)$, $F(x) \in L_2(\mathbb{R})$. An equation of the form (1) with a small coefficient equal to zero, defined on a non-compact interval, is called a degenerate differential equation. A degenerate second-order elliptic equation

$$-\sum_{i,\,j=1}^{n} a_{i\,j}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} = f(x)$$

is known as the stationary Fokker–Planck–Kolmogorov equation. It was studied as a model of Brownian particle motion in the 1920s. Here, a_{ij} is the covariance matrix, and b_i is the bias coefficient.

In what follows, we assume that $p_j(x)$ $(j = 1,2,3)$ is a j -times continuously differentiable positive function. It is known that the operator $L_0 y = y^{(4)} + p_j(x)y^{(j)}$, defined on the set of four-times continuously differentiable and compactly supported functions $C_0^{(4)}(\mathbb{R})$, is closed in the norm of $L_2(\mathbb{R})$. We denote this closure of the operator L_0 by L. An element y $\in D(L)$ satisfying the equality $Ly = F$ is called a solution of equation (1).

A fairly well-studied form of fourth-order differential equations on an unbounded interval is:

$$y^{(4)} + q(x)y = f(x), f \in L_2(\mathbb{R}). \qquad (2)$$

Weak conditions for the unique solvability of equation (2) under the conditions $q \geq \delta > 0$ were obtained in the works of such authors as Ismagilov R.S., Otelbaev M., Boimatov K.Kh., Eweritt W.N., Giertz M. Moreover, in the work of Otelbaev M. it was shown that under certain conditions on the oscillation of the function q, the inequality

$$\left\|y^{(4)}\right\|_2 + \|qy\|_2 \leq C\|f\|_2$$

is satisfied for the solution.

When the last inequality is satisfied, the solution y to equation (2) is called maximally regular. When $q \geq \delta > 0$ and the intermediate coefficients p, r, and s are compactly supported functions, these results obtained for equation (2) also hold for a differential equation of the general form:

$$\mathrm{y}^{(4)} + p(x)\mathrm{y}''' + r(x)\mathrm{y}'' + s(x)\mathrm{y}' + q(x)y = g(x). \qquad (3)$$

This is proved using well-known theorems on small perturbations.

However, many practical problems in the fields of physical processes in turbulent media, biological populations, financial mathematics and stochastic analysis, etc., lead to differential equations of the form (3), in which the intermediate coefficients $p,\ r$ and s are unbounded, and the small coefficient q is zero or does not retain its sign, including the specific differential equation (1). If p_j is an unbounded function, then the methods of the above-mentioned authors, as well as other well-known works, are unsuitable for obtaining conditions for the solvability of equation (1) and the maximal regularity of its solution. This is due to the fact that the operator $p_j \frac{d^j}{dx^j} (j = 1,2,3)$, corresponding to the intermediate term (1), does not obey the fourth-order differential operator $\frac{d^4}{dx^4} + E$ (E is the identity operator). When p_j is a function of general form, this is an open problem. Therefore, the topic of this dissertation, devoted to the problems of solvability and maximal regularity for fourth-order differential equations (1) and some other degenerate differential equations, is undoubtedly relevant from both theoretical and applied points of view.

Objective. To obtain conditions for the unique solvability of degenerate fourth-order binomial differential equations with unbounded coefficients defined on the number line, demonstrate the qualitative properties of their solutions, and prove estimates for the maximal regularity of the solutions.

Research Object. Solvability conditions for fourth-order differential equations with intermediate terms and estimates for the maximal regularity of their solutions.

Research Methods. Methods of local and a priori estimates, certain facts from the theory of closed operators, application of Hardy-type inequalities, identification of adjoint operators, and methods of spectral theory. The nature of the differential equations defined by formula (1) for j = 1, 2, and 3 are different. Various methods are used to study them.

Scientific Novelty. This paper examines the solvability and regularity of degenerate fourth-order binary differential equations in a Hilbert space whose intermediate coefficients can grow without bound. The following new results are obtained:

- It is proven that the presented problems can be reduced to the solvability problem for non-degenerate binary differential equations that preserve the sign of the potential, which have been extensively studied previously.
- Sufficient conditions for p_j for the existence and uniqueness of a solution to differential equation (1) are obtained in each of the cases j = 2, 3.
- In the case j = 1, it is proven that equation (1) has a solution, and it is unique when imposing an additional condition on the oscillation of the coefficient p_j.
- It is shown that the estimate for the maximal regularity of the solution to equation (1) is satisfied in each of the three cases j = 1, 2, 3 when imposing an additional condition on the oscillation of the coefficient p_j.
- Conditions for the correctness of certain fourth-order differential equations defined in divergence form with unbounded leading and intermediate coefficients are obtained.
- Sufficient conditions for the compactness of resolvents of minimal differential operators that form degenerate binomial fourth-order differential equations are presented.

Theoretical and practical significance of the results. The obtained results are of a theoretical nature. Collectively, they expand the theory of solvability and regularity of degenerate fourth-order differential equations. They can be used to evaluate the quality of approximation methods for solutions of general fourth-order equations. They can also be used to solve practical problems that reduce to degenerate fourth-order differential equations.

Relationship of the work with other research projects. This dissertation was completed as part of a state-funded project:

AP14870261 "Qualitative Properties of Solutions of Second-Order Singular Non-Strongly Elliptic Systems."

AP23488049 "Nonlinear Differential Equations with Shifts."

The topic of the dissertation research in the field of "Natural Sciences" corresponds to the priority area "Intellectual Potential of the Country", a specialized scientific area of fundamental and applied research in the fields of mathematics, mechanics, astronomy, physics, chemistry, biology, computer science and geography.

Validation of the obtained results. The results of the work were presented at the International Scientific Conference "Modern Problems of Differential Equations and Their Applications" (Tashkent, 2023), "Traditional International Scientific Conference dedicated to the Day of Scientists (Almaty, 2024)", "Modern Problems of Mathematics, Mechanics and Their Applications (MPMMA)" (Baku, 2024), the 14th International Scientific Conference of the Union of Mathematicians of Georgia (Batumi, 2024), International Scientific Conference "Mathematics and Mathematical Education (ICMME)" (Nevsehir, Turkey, 2024), International Scientific Conference "Non-classical Equations of Mathematical Physics and Their Applications" (Tashkent, 2024), International Scientific Conference "Actual Problems of Analysis, Differential Equations and Algebra (EMJ-2025)" (Astana, 2025), "Traditional International April Mathematical Conference dedicated to the Day of Science of the Republic of Kazakhstan (Almaty, 2025)", XX International Scientific Conference "Lomonosov – 2025" (Astana, 2025).

Publications. The main results of the dissertation are presented in 13 scientific articles and conference proceedings, including one article in a mathematics journal listed in the Scopus database with a CiteScore of at least 25 percentile, four articles in journals recommended by the authorized body, and five articles in foreign journals.

Dissertation Structure. The dissertation consists of an introduction, four sections (each section is divided into subsections), a conclusion, and a bibliography.

Number of references: 64.

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